

‘Fully covariant radiation force on a polarizable particle’ — Reply to the Comment by Volokitin and Persson

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Abstract. We argue that the theories of Volokitin and Persson [Comment arXiv:1405:2525], of Dedkov and Kyasov [*J. Phys.: Condens. Matter* **20** (2008) 354006], and Pieplow and Henkel [*New J. Phys.* **15** (2013) 023027] agree on the electromagnetic force on a small, polarizable particle that is moving parallel to a planar, macroscopic body, as far as the contribution of evanescent waves is concerned. The apparent differences are discussed in detail and explained by choices of units and integral transformations. We point out in particular the role of Lorentz contraction in the procedure used by Volokitin and Persson, where a macroscopic body is ‘diluted’ to get the force on a small particle. Differences that appear in the contribution of propagating photons are briefly mentioned.

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1. Force per particle of Volokitin and Persson

In their Comment [1], Volokitin and Persson (VP) summarize an alternative calculation of the electromagnetic force on a neutral particle moving parallel to a planar half-space. Their approach is ‘macroscopic’ in the sense that the starting point are two half-spaces (1 and 2, say) sliding one against the other with arbitrary velocity [2]. The focus of the present discussion is the lateral force (per unit area) given by a component of the electromagnetic stress tensor, evaluated at the surface of body 1. To arrive at the force between a single, moving particle and a surface, the moving body 2 is ‘diluted’ by taking the limit (notation of VP, cgs units)

$$\epsilon_2(\omega) - 1 \rightarrow 4\pi n_2 \alpha(\omega), \quad |4\pi n_2 \alpha(\omega)| \ll 1 \quad (1)$$

where n_2 is the number density of the constituent atoms (‘particles’ in the following) and $\alpha(\omega)$ their electric polarizability. The resulting force on body 1 (in the frame where it is at rest, while body 2 moves in the x -direction with velocity v) can be written as an integral over electromagnetic waves. The focus of the discussion is the contribution of evanescent waves that takes the form [Eq.(27) of Ref. [1]]

$$\begin{aligned} \text{VP:} \quad f_x^{\text{part, ev}} = & -8\pi\hbar \int_0^\infty \frac{d\omega}{2\pi} \int_{q>\omega/c} \frac{d^2q}{(2\pi)^2} \frac{q_x}{\kappa} e^{-2\kappa z} \text{Im } \alpha(\omega') \\ & \times [N_1(\omega) - N_2(\omega')] \sum_\mu \phi_\mu \text{Im } R_{1\mu} \end{aligned} \quad (2)$$

where $\omega' = \gamma(\omega - q_x v)$ is the frequency of a photon mode in the frame co-moving with the particle. We follow the notation of Ref. [1] except for: k_z is denoted $\kappa = (q^2 - \omega^2/c^2)^{1/2}$;

and the Bose-Einstein distribution is written $N_i(\omega) = \frac{1}{2}[\coth(\hbar\omega/2k_B T_i) - 1]$ ($i = 1, 2$) with T_1 the local temperature of the body at rest, and T_2 the particle's temperature (evaluated in its co-moving frame). The polarization-dependent weight functions and reflection amplitudes are

$$\phi_s = (\omega'/c)^2 + 2\gamma^2\beta^2 q_y^2 \frac{\kappa^2}{q^2}, \quad R_{1s} = \frac{\kappa - \kappa_1}{\kappa + \kappa_1} \quad (3)$$

$$\phi_p = (\omega'/c)^2 + 2\gamma^2(q^2 - \beta^2 q_x^2) \frac{\kappa^2}{q^2}, \quad R_{1p} = \frac{\epsilon_1 \kappa - \kappa_1}{\epsilon_1 \kappa + \kappa_1} \quad (4)$$

where $\beta = v/c$ and the medium propagation constant $\kappa_1 = \sqrt{\kappa^2 - (\epsilon_1 - 1)\omega^2/c^2}$.

2. Comparison to Pieplow and Henkel

Using the fact that $\kappa^2 = q^2 - (\omega/c)^2$, the weight functions ϕ_μ become identical to ours, Eqs.(65, 66) of Ref. [3]. In the sector of evanescent waves, κ is real and positive so that

$$\text{Im} \frac{R_{1\mu} e^{-2\kappa z}}{\kappa} = \frac{e^{-2\kappa z}}{\kappa} \text{Im} R_{1\mu} \quad (5)$$

The evanescent contribution to the friction force by Pieplow and Henkel (PH, Eqs.(67, 69) of Ref. [3]) can therefore be written in the form

$$\begin{aligned} \text{PH:} \quad f_x^{\text{part, ev}} &= \frac{\hbar}{\gamma} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{q>|\omega|/c} \frac{d^2 q}{(2\pi)^2} \frac{q_x}{\kappa} e^{-2\kappa z} \text{Im} \alpha(\omega') \\ &\quad \times [N_1(\omega) - N_2(\omega')] \sum_{\mu} \phi_{\mu} \text{Im} R_{1\mu} \end{aligned} \quad (6)$$

The frequency integral can be reduced to the range $\omega \geq 0$ using the fact that the integrand is even under the transformation $(\omega, q_x) \mapsto (-\omega, -q_x)$: the expressions q_x , $N_1(\omega) - N_2(\omega')$, $\text{Im} \alpha(\omega')$, and $\text{Im} R_{1\mu}$ are all odd under this transformation.* The resulting factor 2 in front of $\int_0^\infty d\omega$ brings Eq.(6) into the form derived by VP, *except* that Eq.(2) contains an additional prefactor $-4\pi\gamma$. We now suggest an explanation for this factor.

The minus sign is due to the fact that VP calculate the force on body 1 (it is dragged along by the moving particle), while PH consider the force on the moving particle (a friction force). Provided the latter is evaluated in the rest frame of body 1 (as done by PH), the two forces are opposite by Newton's *actio = reactio*.

The factor 4π is due to the choice of units: in the cgs units used by VP, the displacement field in the dilute limit of body 2 is given by [see Eq.(1)]

$$\text{VP:} \quad \mathbf{D} = (1 + 4\pi n_2 \alpha) \mathbf{E} \quad (7)$$

* To see this for $N_1(\omega) - N_2(\omega')$, write it as a difference of coth functions. For $\alpha(\omega)$ and $R_{1\mu}(\omega)$, this is a property of Fourier transforms of real-valued response functions. Specifically in $R_{1\mu}$, we use that κ is real and positive for all ω, q_x in the evanescent sector. The medium propagation constant is extended according to $\kappa_1(-\omega) = \kappa_1^*(\omega)$ (real ω), ensuring a retarded solution to the reflection and transmission problem for waves of negative frequencies.

while the same quantity is, in the units used by PH (vacuum permittivity $\varepsilon_0 = 1$),

$$\text{PH :} \quad \mathbf{D} = (1 + n_2 \alpha) \mathbf{E} \quad (8)$$

The factor 4π can therefore be attributed to the different unit for the polarizability.

The factor γ is of course impossible to check by taking the non-relativistic limit. We suggest the following solution based on the ‘dilute medium’ procedure used by VP. The starting point is the lateral stress σ_{xz} on body 1 at rest, a force per unit area. One takes a slice of thickness dz of medium 2 that is centered at a distance z from body 1. This slice increases the force on that body by an amount

$$dF_x = A dz \frac{d\sigma_{xz}}{dz} \quad (9)$$

where A is the area of the body. In the dilute limit, forces are additive so that we convert this into a force per particle (in medium 2) by dividing by the number of particles in that slice

$$f_x^{\text{part}} = \frac{dF_x}{dN_2} = \frac{A dz d\sigma_{xz}/dz}{A n dz} \quad (10)$$

This is the first formula in Eq.(27) of Ref. [1].

The key point is here: n is the number density of body 2 as observed in the rest frame of body 1. This is the only way that an observer fixed to body 1 can define a force per particle. The density n differs from the number density in the co-moving frame due to the Lorentz-Fitzgerald contraction. Hence, we have

$$n = \gamma n_2, \quad f_x^{\text{part}} = \frac{d\sigma_{xz}}{\gamma n_2 dz} \quad (11)$$

where the number density in the co-moving frame is precisely the density n_2 that appears in Eq.(1) above. Indeed, the dielectric response $\epsilon_2(\omega)$ is the one in the rest frame of body 2, as required by the way VP and PH formulate the relativistic description: the field incident on body 2 is transformed into its local rest-frame where $\epsilon_2(\omega)$ can be applied. The Lorentz contraction of the particle density may be the explanation why Eq.(2) is larger by a factor γ compared to Eq.(11).

3. Comparison to Dedkov and Kyasov

In Eq.(13) of Ref. [4], Dedkov and Kyasov (DK) give the following expression for the evanescent contribution to the friction force on a moving particle

$$\begin{aligned} \text{DK:} \quad f_x^{\text{part, ev}} = & \frac{16\pi\hbar}{\gamma} \int_0^\infty \frac{d\omega}{2\pi} \int_{\substack{q_x, q_y \geq 0 \\ q > \omega/c}} \frac{d^2 q}{(2\pi)^2} \frac{q_x}{\kappa} e^{-2\kappa z} \\ & \left\{ \text{Im } \alpha(\omega'_-) \left[N_1(\omega) - N_2(\omega'_-) \right] \sum_\mu \phi_\mu(\omega'_-) \text{Im } R_\mu \right. \\ & \left. - (\omega'_- \mapsto \omega'_+) \right\} \end{aligned} \quad (12)$$

where $\omega'_\pm = \gamma(\omega \pm q_x v)$. We have used the translation Table 1 for the transcription into the notation of VP (except for κ and $N_i(\omega)$ as mentioned after Eq.(2)). Note that for a fair

comparison, we have neglected the contribution from the magnetic polarizability α_m and written $\alpha_e = \alpha$. Eq.(12) uses an integration range over only one quadrant in the \mathbf{q} -plane.

	temperatures		photon modes				occupation	polarization weights		
DK [4]	T_1	T_2	\mathbf{k}	q_0	\tilde{q}_0	$\gamma\omega^\pm$	$W(\omega/T_2, \omega^\pm\gamma/T_1)$	$\gamma^2\chi_e^{(\pm)}(\omega, \mathbf{k})$	Δ_e	Δ_m
VP [1]	T_2	T_1	\mathbf{q}	k_z, κ	q_z	ω'_\pm	$2[N_1(\omega) - N_2(\omega'_\pm)]$	$\phi_p(\omega'_\pm)$	R_{1p}	R_{1s}

Table 1. Translated notations from Dedkov and Kyasov [4] to Volokitin and Persson [1].

Since the integrand is even in q_y , a prefactor 2 can be removed and the integral extended over the entire q_y -axis (restricted to evanescent waves, of course). The two lines in Eq.(12) involving $\omega'_- = \omega'$ and ω'_+ only differ by the sign of q_x and can therefore be combined into one integral over the q_x -axis (in the evanescent sector). After these manipulations, we arrive at Eq.(2), except for the factor $-1/\gamma$. The minus sign is explained as above. If one includes the Lorentz-contracted density in the procedure for taking the dilute limit, as outlined above, the formulas by VP and by DK may be brought into full agreement.

4. Propagating sector

VP do not discuss in their Comment the contribution from propagating photons. A quick glance at their Eq.(22), first term, suggests that the ‘dilution procedure’ gives a result that is qualitatively different. The rules spelled out after Eq.(26) give to leading order a contribution to the stress (force per area) on body 1 given by

$$\sigma_{xz}^{\text{pr}} = -\hbar \int_0^\infty \frac{d\omega}{2\pi} \int_{q \leq \omega/c} \frac{d^2q}{(2\pi)^2} q_x (2 - |R_{1p}|^2 - |R_{1s}|^2) [N_1(\omega) - N_2(\omega')] \quad (13)$$

Note that does not allow for a dilute limit because it is not proportional to the density n_2 . (It only depends on the temperature T_2 of the diluted body 2.) A detailed comparison to the result given by our approach would go beyond the purpose of this Reply, as there are also physical reasons to expect a difference.*

Let us compare in the following the results of DK and PH in the propagating sector. Eq.(13) of Ref. [4] by DK provides an integral representation whose first line actually corresponds to a free-space (fs) contribution (taking only the electric polarizability)

$$\text{DK:} \quad f_x^{\text{part,pr}} \Big|_{\text{fs}} = -\frac{4\hbar\gamma}{c^4} \int_0^\infty \frac{d\omega}{2\pi} \omega^4 \int_{-1}^1 dx x (1 + \beta x)^2 \times \text{Im} \alpha(\omega_1) [N_1(\omega) - N_2(\omega_1)] \quad (14)$$

where $\omega_1 = \gamma\omega(1 + \beta x)$. The force in free space, filled with blackbody radiation at temperature T_1 , is apparent from Eq.(56) in PH’s Ref. [3]. Eqs.(52, 54) in that paper translate

* For example, an infinitely thick half-space does not show any transmission for radiation emitted by body 1, while a single particle does. The expression $1 - |R'_{2\mu}|^2$ gives the absorption of a half-space and appears in the analogue of Eq.(19) of Ref. [1], first line, to calculate the emission from body 2. If body 2 were a thin layer, however, also its transmission would appear here, and even become significant in the dilute limit.

into the present notation as follows

$$\text{PH:} \quad f_x^{\text{part,pr}} \Big|_{\text{fs}} = \frac{2\hbar\gamma}{\pi c^3} \int_0^\infty \frac{d\omega}{2\pi} \int \frac{d\Omega}{4\pi} \omega q_x (\omega - \beta q_x)^2 \times \text{Im } \alpha(\omega') [N_1(\omega) - N_2(\omega')] \quad (15)$$

where the symmetry manipulations mentioned after Eq.(6) have been used for the ω -integral. We integrate over the directions of photon wave vectors (solid angle $d\Omega$), their length being fixed to ω/c . By rotational symmetry around the x -axis, this integral can be reduced to (substitution $q_x = (\omega/c)x$)

$$\text{PH:} \quad f_x^{\text{part,pr}} \Big|_{\text{fs}} = \frac{\hbar\gamma}{\pi c^4} \int_0^\infty \frac{d\omega}{2\pi} \omega^4 \int_{-1}^1 dx x (1 - \beta x)^2 \times \text{Im } \alpha(\omega') [N_1(\omega) - N_2(\omega')] \quad (16)$$

where now $\omega' = \gamma\omega(1 - \beta x)$. Flipping the sign of x , we arrive at Eq.(14), up to a factor 4π that arises again from the choice of units for the polarizability (see above).

The surface-dependent part of DK, Eq.(13) in Ref. [4], involves the reflection coefficients and reads (see translation Table 1):

$$\begin{aligned} \text{DK:} \quad f_x^{\text{part,pr}} \Big|_{\text{surf}} &= \frac{16\pi\hbar}{\gamma} \int_0^\infty \frac{d\omega}{2\pi} \int_{\substack{q_x, q_y \geq 0 \\ q \leq \omega/c}} \frac{d^2q}{(2\pi)^2} \frac{q_x}{q_z} (-\sin 2q_z z) \\ &\times \left\{ \text{Im } \alpha(\omega'_-) [N_1(\omega) - N_2(\omega'_-)] \sum_{\mu} \phi_{\mu}(\omega'_-) \text{Im } R_{1\mu} \right. \\ &\quad \left. - (\omega'_- \mapsto \omega'_+) \right\} \\ &+ \frac{16\pi\hbar}{\gamma} \int_0^\infty \frac{d\omega}{2\pi} \int_{\substack{q_x, q_y \geq 0 \\ q \leq \omega/c}} \frac{d^2q}{(2\pi)^2} \frac{q_x}{q_z} \cos(2q_z z) \\ &\times \left\{ \text{Im } \alpha(\omega'_-) [N_1(\omega) - N_2(\omega'_-)] \sum_{\mu} \phi_{\mu}(\omega'_-) \text{Re } R_{1\mu} \right. \\ &\quad \left. - (\omega'_- \mapsto \omega'_+) \right\} \end{aligned} \quad (17)$$

where the prescription $\{R_e^{\pm}, R_m^{\pm} \rightarrow \tilde{R}_e^{\pm}, \tilde{R}_m^{\pm}\}$ has been applied as explained after Eq.(25) of Ref. [4]. We have used the notation $q_z = (\omega/c)[1 - (cq/\omega)^2]^{1/2}$ which is real. We extend the \mathbf{q} integral from one quadrant to the entire circle $q \leq \omega/c$ using the manipulations described after Eq.(12), and get

$$\begin{aligned} \text{DK:} \quad f_x^{\text{part,pr}} \Big|_{\text{surf}} &= \frac{8\pi\hbar}{\gamma} \int_0^\infty \frac{d\omega}{2\pi} \int_{q \leq \omega/c} \frac{d^2q}{(2\pi)^2} \frac{q_x}{q_z} \text{Im } \alpha(\omega'_-) \{ [N_1(\omega) - N_2(\omega'_-)] \\ &\times \sum_{\mu} \phi_{\mu}(\omega'_-) (\text{Re } R_{1\mu} \cos 2q_z z - \text{Im } R_{1\mu} \sin 2q_z z) \} \end{aligned} \quad (18)$$

The result of PH can be found from Eqs.(67, 69) in Ref. [3] and is an integral identical to Eq.(6), with the \mathbf{q} -range restricted to $q \leq |\omega|/c$ (propagating waves) and the replacement

$$\frac{e^{-2\kappa z}}{\kappa} \text{Im } R_{1\mu} \mapsto \frac{\text{Re}(R_{1\mu} e^{2iq_z z})}{q_z} \quad (19)$$

recalling that q_z is real. The manipulations mentioned after Eq.(6) bring this expression to a positive-frequency integral of the form

$$\begin{aligned} \text{PH: } f_x^{\text{part,pr}} \Big|_{\text{surf}} &= \frac{2\hbar}{\gamma} \int_0^\infty \frac{d\omega}{2\pi} \int_{q \leq \omega/c} \frac{d^2 q}{(2\pi)^2} \frac{q_x}{q_z} \text{Im } \alpha(\omega') [N_1(\omega) - N_2(\omega')] \\ &\times \sum_\mu \phi_\mu (\text{Re } R_{1\mu} \cos 2q_z z - \text{Im } R_{1\mu} \sin 2q_z z) \end{aligned} \quad (20)$$

Up to the familiar 4π , this is identical to Eq.(18) because $\omega'_- = \omega'$.

5. Discussion

The word ‘covariant’ in the title of our paper [3] may have led to the impression that this is the only way to formulate a fully relativistic theory. This is of course wrong: it is just a convenient formulation, and other approaches, that do not work with 4-vectors and metric tensors etc., give equally valid results, even for relativistic velocities. The calculations of VP [2] and DK [4] are examples of these. The advantage of the ‘manifestly covariant’ formulation is that transformation properties are relatively easy to identify. For example, the transformation properties of the electromagnetic field and the polarization field both arise from tensor fields, namely $F_{\mu\nu}$ and $M^{\mu\nu}$.

We have provided some technical details to show that DK and PH get the same electromagnetic force for the particle+surface scenario, as mentioned in [3]. The agreement holds for both propagating and evanescent waves and for arbitrary temperatures. The approach of VP apparently differs by a factor $-4\pi\gamma$ for evanescent waves. We have argued that this factor disappears when the same units are used and when the procedure of diluting the moving body takes into account the relativistic contraction of densities.

For propagating photons, a disagreement between VP and PH arises. We have argued that it is not obvious how to combine thermal equilibrium in a medium with the dilution procedure: indeed, as long as body 2 is infinitely thick, there can be no contribution ‘from its back side’ to the electromagnetic stress between bodies 1 and 2. It is well possible that a calculation where body 2 is a slab of finite thickness which is then diluted, will retrieve the particle+surface case in full, provided the photons incident on the ‘back side’ of the slab are in equilibrium in the same frame and temperature as body 1. Otherwise a drag stress must be expected on body 1, similar to the force on a particle that moves relative to the frame where a thermal radiation field is in equilibrium [5].

- [1] Volokitin A I and Persson B N J 2014 Comment on “fully covariant radiation force on a polarizable particle” arXiv:1405.2525
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- [5] Einstein A 1917 *Physik. Zeitschr.* **18** 121–28